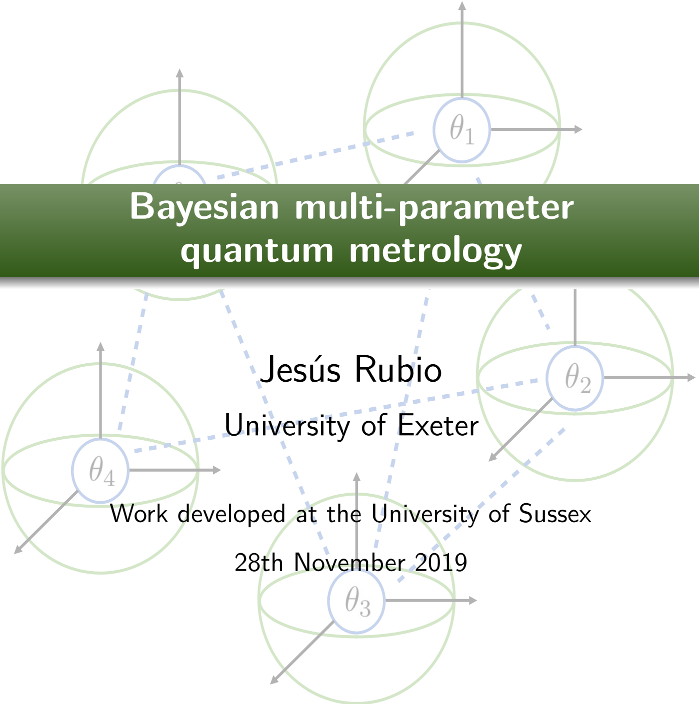
Bayesian multi-parameter quantum metrology



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Outline

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Summary and conclusions

Bayesian probabilities

Definition and calculus of probabilities

$P(A|B) \equiv$ real number representing the degree of plausibility for A to be true given B, where A and B are *propositions*.

1. $0 \leqslant P(A|B) \leqslant 1$,
2. $P(A|B) = 1$ when it can be concluded that A is true on the basis of B,
3. $P(\neg A|B) = 1 - P(A|B)$, and
4. $P(A \wedge B|C) = P(A|C)P(B|A \wedge C)$,

[1] R. T. Cox, American Journal of Physics, 14(1):113, 1946.

[2] E. T. Jaynes, Cambridge University Press, 2003.

[3] L. E. Ballentine, Foundations of Physics, 46, 2016.

Bayes theorem

$$P(A|B \wedge I_0) = \frac{P(A|I_0)P(B|A \wedge I_0)}{P(B|I_0)},$$

where

- $P(A|I_0) \equiv$ prior probability,
- $P(B|A \wedge I_0) \equiv$ likelihood,
- $P(A|B \wedge I_0) \equiv$ posterior probability, and
- $P(B|I_0) \equiv$ normalisation constant.

The prior $P(A|I_0)$ is updated using the new information about A provided by the evidence B, which is encoded in the likelihood $P(B|A \wedge I_0)$, and the overall result is the construction of the posterior $P(A|B \wedge I_0)$.

An example

- $I_1 =$ Let θ be the magnitude of an optical phase $\implies 0 \leqslant \theta < 2\pi$.
- $I_2 =$ We are completely ignorant about such magnitude

  $\implies$ the estimation problems associated with θ and $\theta' = \theta + c$, for some constant c and taking it to be modulo 2π, are equivalent.

We would like to construct the probability $P(d\theta|I_0) = p(\theta)d\theta$, with $I_0 = I_1 \wedge I_2$. From I_2 we can derive the *functional equation*

$$p(\theta)d\theta = p(\theta')d\theta' = p(\theta + c)d\theta \implies p(\theta) = p(\theta + c),$$

whose solution is $p(\theta) \propto 1$, and upon its normalisation using I_1 we conclude that

$$p(\theta) = 1/(2\pi), \text{ for } 0 \leqslant \theta < 2\pi.$$

[2] E. T. Jaynes, Cambridge University Press, 2003.

Quantum estimation theory

Uncertainty and estimation

- We wish to estimate d unknown parameters $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$ by constructing the estimators $\boldsymbol{g}(\boldsymbol{m}) = (g_1(\boldsymbol{m}), \ldots, g_d(\boldsymbol{m}))$ from the μ experimental outcomes $\boldsymbol{m} = (m_1, \ldots, m_\mu)$.

- $\mathcal{D}[\boldsymbol{g}(\boldsymbol{m}), \boldsymbol{\theta}] \equiv$ deviation function quantifying the deviation of our estimates $\boldsymbol{g}(\boldsymbol{m})$ when the parameters happened to be $\boldsymbol{\theta}$.

- In a theoretical study we know neither the parameters nor the experimental outcomes; consequently, an appropriate measure of uncertainty is

$$\bar{\epsilon} = \int d\boldsymbol{\theta} d\boldsymbol{m} \; p(\boldsymbol{\theta}, \boldsymbol{m}) \; \mathcal{D}[\boldsymbol{g}(\boldsymbol{m}), \boldsymbol{\theta}].$$

The quantum part of the problem

Using the product rule and the Born rule we can express the joint probability as

$$p(\boldsymbol{m}, \boldsymbol{\theta}) = p(\boldsymbol{\theta})p(\boldsymbol{m}|\boldsymbol{\theta}) = p(\boldsymbol{\theta})\mathrm{Tr}\left[E(\boldsymbol{m})\varrho(\boldsymbol{\theta})\right],$$

where

- $p(\boldsymbol{\theta}) \equiv$ prior probability,
- $\varrho_0 \rightarrow \varrho(\boldsymbol{\theta}) \equiv$ state after encoding the unknown parameters $\boldsymbol{\theta}$, and
- $E(\boldsymbol{m}) \equiv$ probability-operator measurement (POM) with outcomes $\boldsymbol{m}$.

Thus

$$\bar{\epsilon} = \int d\boldsymbol{\theta} d\boldsymbol{m}\; p(\boldsymbol{\theta})\mathrm{Tr}\left[E(\boldsymbol{m})\varrho(\boldsymbol{\theta})\right]\mathcal{D}[\boldsymbol{g}(\boldsymbol{m}), \boldsymbol{\theta}].$$

The fundamental equations of the optimal quantum strategy

Using $E(\boldsymbol{g}) = \int d\boldsymbol{m}\, \delta\left(\boldsymbol{g}(\boldsymbol{m}) - \boldsymbol{g}\right) E(\boldsymbol{m})$ we can recast the uncertainty as

$$\bar{\epsilon} = \int d\boldsymbol{g}\ \mathrm{Tr}\left[E(\boldsymbol{g})Q(\boldsymbol{g})\right], \text{ with } Q(\boldsymbol{g}) = \int d\boldsymbol{\theta}\ p(\boldsymbol{\theta})\varrho(\boldsymbol{\theta})\mathcal{D}(\boldsymbol{g},\boldsymbol{\theta}).$$

If $E_{\mathrm{opt}}(\boldsymbol{g})$ is the *optimal strategy*, then there exists a Hermitian operator Y satisfying that

$$\begin{cases} Y = \int d\boldsymbol{g}\ Q(\boldsymbol{g})E_{\mathrm{opt}}(\boldsymbol{g}) = \int d\boldsymbol{g} E_{\mathrm{opt}}(\boldsymbol{g})Q(\boldsymbol{g}), \\ Q(\boldsymbol{g}) - Y \geqslant 0, \end{cases}$$

and we have that $\bar{\epsilon} \geqslant \bar{\epsilon}_{\mathrm{min}} = \mathrm{Tr}(Y)$.

[4] A. S. Holevo, Proc. of the 2nd Japan-USSR Symp. on Prob. Theory, 104119, 1973.

[5] C. W. Helstrom, Academic Press, New York, 1976.

[6] R. Demkowicz-Dobrzański *et al.*, Progress in Optics, 60:345435, 2015.

Multi-parameter shot-by-shot methodology

Practical aspects of our problem: moderate prior knowledge

Suppose we know that the parameters are localised within a hypervolume Δ_0 centred around $\bar{\boldsymbol{\theta}}$, so that the flat prior $p(\boldsymbol{\theta}) = 1/\Delta_0$ is appropriate in that region.

Intermediate prior information regime: neither $\Delta_0 \rightarrow 0$ nor $\Delta_0 \gg 1$.

In that case,

$$\bar{\epsilon} \approx \bar{\epsilon}_{\mathrm{mse}} = \sum_{i=1}^{d} w_i \int d\boldsymbol{\theta} d\boldsymbol{m}\; p(\boldsymbol{\theta}, \boldsymbol{m}) \left[g_i(\boldsymbol{m}) - \theta_i\right]^2$$

where $w_i \geqslant 0$ is the relative importance of estimating θ_i and $\sum_{i=1}^{d} w_i = 1$.

[7] Jesús Rubio *et al.*, J. Phys. Comm., 2(1):015027 (2018).

[8] T. J. Proctor *et al.*, arXiv:1702.04271, 2017.

Practical aspects of our problem: repetitions

If the operations $\rho_0 \rightarrow \rho(\boldsymbol{\theta}) = U(\boldsymbol{\theta})\rho_0 U^{\dagger}(\boldsymbol{\theta}) \rightarrow E(m_i)$ are repeated μ times, then

1) $\varrho(\boldsymbol{\theta}) = \underbrace{\rho(\boldsymbol{\theta}) \otimes \cdots \otimes \rho(\boldsymbol{\theta})}_{\mu \text{ times}}$

2) $E(\boldsymbol{m}) = E(m_1) \otimes \cdots \otimes E(m_\mu)$

However, if we try to solve Helstrom and Holevo's equations using the state in 1), then the optimal strategy may involve *collective measurements*, which cannot be written as 2).

Instead, **let us focus first on the single-shot case**:

$$\bar{\epsilon}_{\mathrm{mse}} = \sum_{i=1}^{d} w_i \int d\boldsymbol{\theta} dm \; p(\boldsymbol{\theta}, m) \left[g_i(m) - \theta_i\right]^2 .$$

New Bayesian multi-parameter bound

The error can be rewritten as $\bar{\epsilon}_{\mathrm{mse}} = \mathrm{Tr}[W_D \Sigma_{\mathrm{mse}}]$, where $W_D = \mathrm{diag}(w_1, \dots, w_d)$ and

$$\Sigma_{\mathrm{mse}} = \int d\boldsymbol{\theta} dm \; p(\boldsymbol{\theta}, m) \left[\boldsymbol{g}(m) - \boldsymbol{\theta}\right] \left[\boldsymbol{g}(m) - \boldsymbol{\theta}\right]^{\intercal}.$$

In addition, we can construct the scalar quantity

$$\boldsymbol{u}^{\intercal} \Sigma_{\mathrm{mse}} \boldsymbol{u} = \int d\boldsymbol{\theta} dm \; p(\boldsymbol{\theta}, m) \left[g_u(m) - \theta_u\right]^2,$$

with

$$g_u(m) = \boldsymbol{u}^{\intercal} \boldsymbol{g}(m) = \boldsymbol{g}^{\intercal}(m) \, \boldsymbol{u},$$

$$\theta_u = \boldsymbol{u}^{\intercal} \boldsymbol{\theta} = \boldsymbol{\theta}^{\intercal} \boldsymbol{u},$$

and $\boldsymbol{u}$ being an arbitrary real vector.

Applying Helstrom and Holevo's equations to such scalar quantity we find that

$$\boldsymbol{u}^{\intercal}\Sigma_{\mathrm{mse}}\boldsymbol{u} \geqslant \int d\boldsymbol{\theta}\ p(\boldsymbol{\theta})\theta_u^2 - \mathrm{Tr}(\rho S_u^2),$$

where

$$\rho = \int d\boldsymbol{\theta}\ p(\boldsymbol{\theta})\rho(\boldsymbol{\theta}), S_u\rho + \rho S_u = 2\bar{\rho}_u,\ \text{and}\ \bar{\rho} = \int d\boldsymbol{\theta}\ p(\boldsymbol{\theta})\rho(\boldsymbol{\theta})\boldsymbol{\theta}.$$

In addition, recalling that $\theta_u = \sum_{i=1}^{d} u_i\theta_i$, note that

$$\bar{\rho}_u = \sum_{i=1}^{d} u_i\bar{\rho}_i,\ \text{with}\ \bar{\rho}_i = \int d\boldsymbol{\theta} p(\boldsymbol{\theta})\rho(\boldsymbol{\theta})\theta_i;$$

$$S_u = \sum_{i=1}^{d} u_i S_i,\ \text{with}\ S_i\rho + \rho S_i = 2\bar{\rho}_i\ \text{and}\ S_i\ \text{Hermitian}.$$

[5] C. W. Helstrom, Academic Press, New York, 1976.

Finally, imposing that the previous inequality is true for all $\boldsymbol{u}$ we arrive at the *matrix quantum bound*

$$\Sigma_{\text{mse}} \geqslant \Sigma_q = \int d\boldsymbol{\theta} p(\boldsymbol{\theta}) \boldsymbol{\theta}\boldsymbol{\theta}^{\mathsf{T}} - \mathcal{K},$$

where

$$\mathcal{K}_{ij} = \mathrm{Tr}\left[\rho\left(S_i S_j + S_j S_i\right)\right]/2.$$

Moreover, its scalar version reads as

$$\bar{\epsilon}_{\text{mse}} \geqslant \sum_{i=1}^{d} w_i (\Delta\theta_{p,i}^2 - \Delta S_{\rho,i}^2),$$

with $\Delta S_{\rho,i}^2 \equiv \mathrm{Tr}(\rho S_i^2) - \mathrm{Tr}(\rho S_i)^2$ and $\Delta\theta_{p,i}^2 \equiv \int d\boldsymbol{\theta} p(\boldsymbol{\theta}) \theta_i^2 - [\int d\boldsymbol{\theta} p(\boldsymbol{\theta}) \theta_i]^2$. This is our central result.

[9] Jesús Rubio and J. Dunningham, arXiv:1906.04123, 2019.

Saturability and shot-by-shot method

Our bound can be saturated when $\boldsymbol{g}(\boldsymbol{m}) = \int d\boldsymbol{\theta}\; p(\boldsymbol{\theta}|\boldsymbol{m})\boldsymbol{\theta}$ and $[S_i, S_j] = 0$ for all i, j. In that case, the *optimal measurement* is given by the projections on the common eigenstates of $\{S_i\}$.

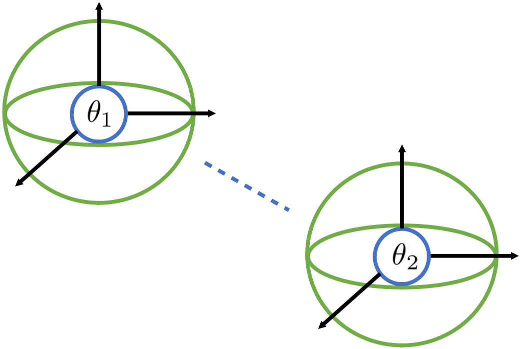
If the optimal single-shot POM $E(m_i) \equiv E(s_i) = |\psi(s_i)\rangle\langle\psi(s_i)|$, with outcome s_i, exists, then the uncertainty associated with μ repetitions of the *optimal strategy* (estimator + POM) is

$$\bar{\epsilon}_{\mathrm{mse}} = \sum_{i=1}^{d} w_i \int d\boldsymbol{\theta} d\boldsymbol{s}\; p(\boldsymbol{\theta}, \boldsymbol{s}) \left[g_i(\boldsymbol{s}) - \theta_i\right]^2 .$$

[9] Jesús Rubio and J. Dunningham, arXiv:1906.04123, 2019.

Quantum sensing networks

Qubit network

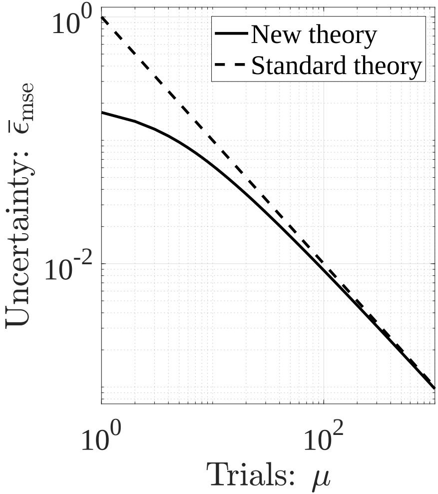


- Network initial state: $|\psi_0\rangle = [|00\rangle + \gamma(|01\rangle + |10\rangle) + |11\rangle]/\sqrt{2(1+\gamma^2)}$.
- Unitary encoding: $U(\theta_1, \theta_2) = \exp[-i(\sigma_{z,1}\theta_1 + \sigma_{z,2}\theta_2)/2]$.
- Prior: $p(\theta_1, \theta_2) = 4/\pi^2$, when $(\theta_1, \theta_1) \in [-\pi/4, \pi/4] \times [-\pi/4, \pi/4]$.
- Weighting matrix: $W_D = \mathbb{I}/2$.
- It may be shown that its single-shot minimum is achieved for $|\gamma| = 1$, which is a *local strategy*.
- In that case we have that

$$S_1 = \frac{(4-\pi)}{\pi\sqrt{2}}\sigma_y \otimes \mathbb{I},\ S_2 = \frac{(4-\pi)}{\pi\sqrt{2}}\mathbb{I} \otimes \sigma_y,$$

which commute. Thus S_1 and S_2 are the *optimal quantum estimators*, and by diagonalising them we find the optimal single-shot POM $|s_+, s_+\rangle$, $|s_-, s_-\rangle$, $|s_+, s_-\rangle$, $|s_-, s_+\rangle$, where $|s_\pm\rangle = (|0\rangle \pm i\,|1\rangle)/\sqrt{2}$.

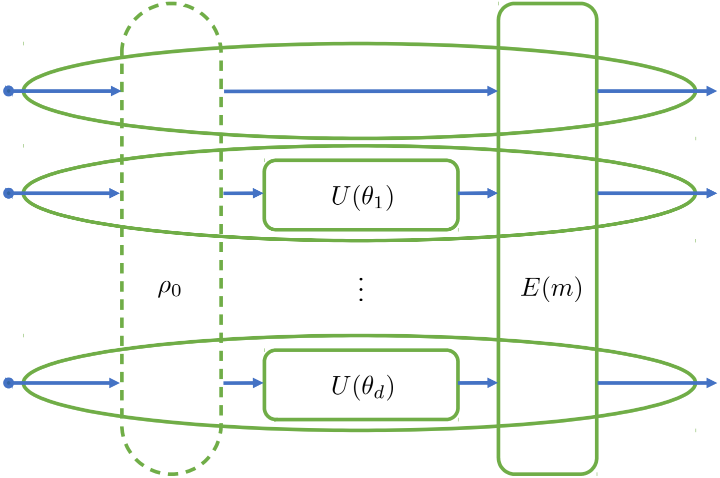
[9] Jesús Rubio and J. Dunningham, arXiv:1906.04123, 2019.



[8] T. J. Proctor *et al.*, arXiv:1702.04271, 2017.

[9] Jesús Rubio and J. Dunningham, arXiv:1906.04123, 2019.

Discrete phase imaging



- Network state: $|\psi_0\rangle = \frac{1}{\sqrt{d+\alpha^2}}(\alpha\,|\bar{n}_0\rangle + \sum_{k=1}^{d} |\bar{n}_k\rangle)$.
- Unitary encoding: $U(\theta_j) = \exp(-i a_j^\dagger a_j \theta_j)$, for $1 \leqslant j \leqslant d$. The remaining mode is employed as a reference.
- Prior: $\Delta_0 = (2\pi/\bar{n})^d$, with $\bar{n} \geqslant 4$, $\bar{\boldsymbol{\theta}} = (0, 0, \dots)$.
- Weighting matrix: $W_D = \mathbb{I}/d$.
- The minimum single-shot uncertainty is found to be

$$\bar{\epsilon}_{\rm mse} \geqslant \frac{1}{\bar{n}^2}\left[\frac{\pi^2}{3} - \frac{4}{(1+\sqrt{d})^2}\right] \underset{d \gg 1}{\longrightarrow} \frac{1}{\bar{n}^2}\left(\frac{\pi^2}{3} - \frac{4}{d}\right).$$

- However,

$$S_k = \frac{-2i\alpha}{\bar{n}\,(1+\alpha^2)}\left(|\bar{n}_k\rangle\langle\bar{n}_0| - |\bar{n}_0\rangle\langle\bar{n}_k|\right),$$

  so that $[S_k, S_j] \neq 0$. Does this mean that we cannot reach the scaling with d predicted by our bound?

- In a local protocol $\rho_0 = \rho_0^{\mathrm{ref}} \otimes \rho_0^{(1)} \otimes \cdots \otimes \rho_0^{(d)}$, with $\rho_0^{(k)} = |\phi_0^{(k)}\rangle\langle\phi_0^{(k)}|$ in the pure case, we have that $S_k = \mathbb{I}_{\mathrm{ref}} \otimes \mathbb{I} \otimes \cdots \otimes S^{(k)} \otimes \cdots \otimes \mathbb{I}$, which commute trivially with each other.

- If we choose

$$|\phi_0\rangle = \left[\sqrt{1 - \frac{\bar{n}}{N(d+1)}}\,|0\rangle + \sqrt{\frac{\bar{n}}{N(d+1)}}\,|N\rangle\right],$$

with $N = \bar{n}$, then we find that

$$\bar{\epsilon}_{\mathrm{mse}} \geqslant \frac{1}{\bar{n}^2}\left[\frac{\pi^2}{3} - \frac{4d}{(1+d)^2}\right] \underset{d \gg 1}{\longrightarrow} \frac{1}{\bar{n}^2}\left(\frac{\pi^2}{3} - \frac{4}{d}\right),$$

which provides the same scaling that the global scheme does; consequently, the shot-by-shot method could be applied even when our bound cannot be saturated for the global strategy.

[9] Jesús Rubio and J. Dunningham, arXiv:1906.04123, 2019.

Summary and conclusions

- We have reviewed the Bayesian approach and the fundamental equations for the optimal quantum strategy.
- To solve realistic problems with limited data and moderate prior knowledge, we have derived a new single-shot Bayesian quantum bound, and we have exploited it to construct a multi-parameter shot-by-shot methodology.
- Among all the bounds that neglect the interference between individually optimal quantum strategies, our result is arguably the preferred option, since it recovers the true optimum in the limit of a single parameter, and it gives the true multi-parameter optimum when $\{S_i\}$ commute.

- We have applied these ideas to a qubit network and a model for discrete phase imaging, and we have shown that entanglement is not always required to achieved the optimal precision in the regime of limited data.

- In summary, our method provides a powerful and novel framework to study schemes with limited data and moderate prior knowledge, a regime of practical interest and normally out of the scope of other techniques in the literature. If you are interested in learning more about this approach to quantum metrology, please have a look at

  Jesús Rubio and J. Dunningham, New J. of Phys., 21(4):043037, 2019.

  Jesús Rubio and J. Dunningham, arXiv:1906.04123, 2019.

  Moreover, my PhD thesis will be released soon. Stay tuned!

Thank you for your attention!