Bayesian quantum metrology for time estimation

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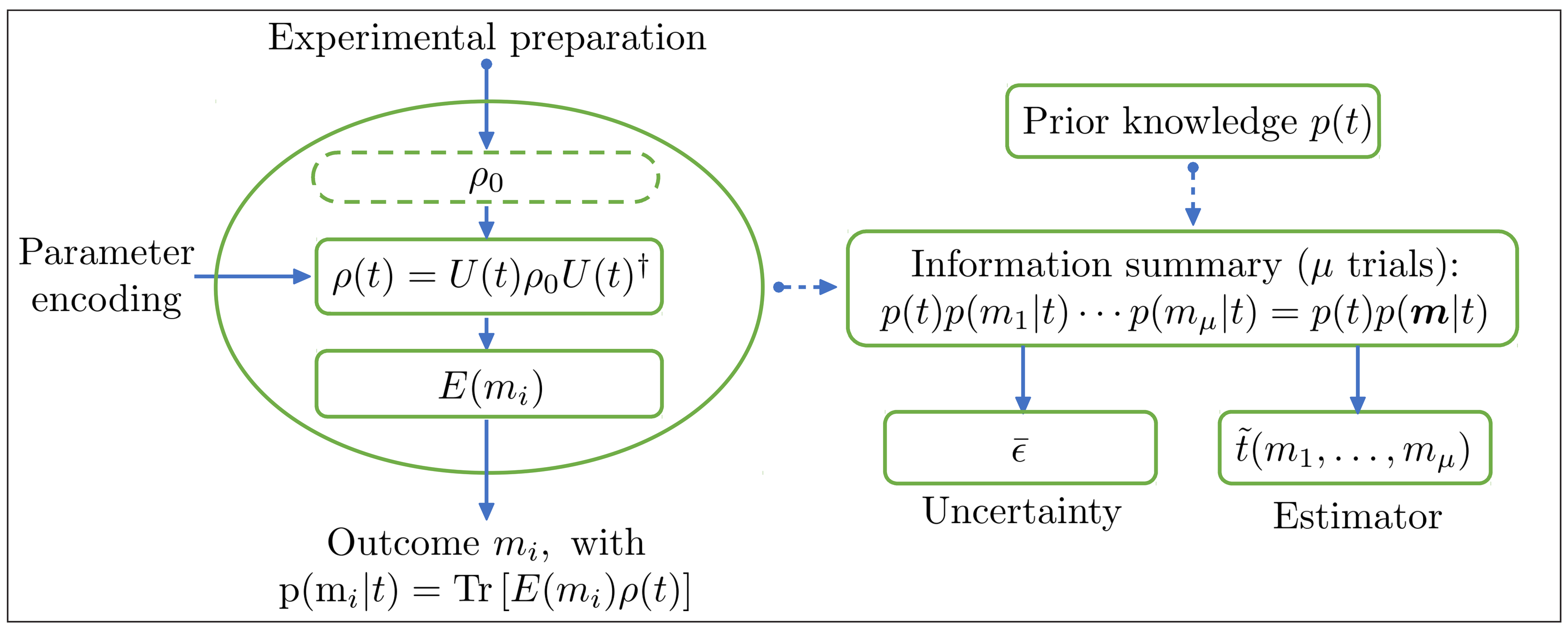
Research problem

We address the problem of estimating the elapsed time from the evolution of a quantum system using the Bayesian version of quantum metrology, which is a framework providing the tools to study protocols with a finite amount of resources when certain prior information is available. Furthermore, we compare the results derived from this approach with those that are predicted by the asymptotic theory in terms of the Fisher information and the Cramér-Rao bound. Our results constitute a proof of concept that could be useful in the study of more realistic clocks, and from a fundamental perspective they offer a way of quantifying time by means of a Bayesian uncertainty inequality that can take into account any prior information that we may have.

Bayesian quantum metrology: methodology

Quantum metrology protocols



Measure of uncertainty

Given μ identical and independent trials, we quantify the estimation uncertainty with

$$\bar{\epsilon}_{\mathrm{mse}}(\mu) = \int d\boldsymbol{m}dt\, p(t)p(\boldsymbol{m}|t)\left[\tilde{t}(\boldsymbol{m}) - t\right]^2.$$

Asymptotic uncertainty

For pure states and a unitary $U(t) = \exp(-i\mathcal{G}t)$ with generator $\mathcal{G}$, the quantum Fisher information is

$$F_q = 4[\mathrm{Tr}(\rho_0\mathcal{G}^2) - \mathrm{Tr}(\rho_0\mathcal{G})^2].$$

If $\mu \gg 1$ and a moderate amount of prior knowledge is available, then [1]

$$\bar{\epsilon}_{\mathrm{mse}}(\mu \gg 1) \gtrsim 1/(\mu F_q)$$

The asymptotic theory is a useful guide to select probes with great sensitivity.

Optimal single-shot uncertainty

For a single shot we find that [2]

$$\bar{\epsilon}_{\mathrm{mse}}(\mu = 1) \geqslant \Delta t_p^2 - \Delta\tilde{T}_\rho^2,$$

with

- $\Delta t_p^2 = \langle (t - \langle t \rangle)^2 \rangle$, where we use the notation $\langle \square \rangle = \int dt\, p(t) \square$;
- $\Delta\tilde{T}_\rho^2 = \mathrm{Tr}(\rho\tilde{T}^2) - \mathrm{Tr}(\rho\tilde{T})^2$; and
- $\tilde{T}$ is the optimal quantum estimator defined by $\tilde{T}\rho + \rho\tilde{T} = 2\bar{\rho}$, $\rho = \langle \rho(t) \rangle$ and $\bar{\rho} = \langle \rho(t)t \rangle$ [3,4].

The bound is saturated by measuring $\tilde{T} = \int ds\, s\, |s\rangle\langle s|$, that is, when $E(s) = |s\rangle\langle s|$.

Our strategy (more in [1,2])

Given $p(t)$ and $\mathcal{G}$,

1. Choose a ρ_0 with maximum F_q.
2. Select the POVM elements $\{E(s)\}$ that are optimal for both $\mu = 1$ and $\mu \gg 1$.
3. Verify that the prior information is sufficient for the selected scheme.

Then we can study the uncertainty

$$\bar{\epsilon}_{\mathrm{mse}}(\mu) = \Delta t_p^2 - \Delta\tilde{t}^2$$

associated with μ identical and independent trials, where

- $\Delta\tilde{t}^2 = \int d\boldsymbol{s}\, p(\boldsymbol{s})[\tilde{t}(\boldsymbol{s}) - \langle t \rangle]^2$,
- $\tilde{t}(\boldsymbol{s}) = \int dt\, p(t|\boldsymbol{s})t$ is the optimal estimator,
- $p(t|\boldsymbol{s}) \propto p(t)p(\boldsymbol{s}|t)$ is the posterior probability,
- $p(\boldsymbol{s}) = \int dt\, p(t)p(\boldsymbol{s}|t)$, and
- $\boldsymbol{s} = (s_1, \ldots, s_\mu)$.

Therefore, our measurement strategy is optimal both for a single shot and in the asymptotic regime of many trials. Moreover, this approach is also optimal in the regime of limited data for experiments where we cannot or do not wish to correlate different trials.

The qubit clock I: System configuration

Qubit clock

1. *Initial state*: $\rho_0 = (\mathbb{I} + \hat{\boldsymbol{r}}^\intercal \cdot \boldsymbol{\sigma})/2$
2. *Generator*: $\mathcal{G} = (E/\hbar)\hat{\boldsymbol{n}}^\intercal \cdot \boldsymbol{\sigma}$

State selection and time evolution

The quantum Fisher information is

$$F_q = 4(\hbar/E)^2[1 - (\hat{\boldsymbol{r}}^\intercal \cdot \hat{\boldsymbol{n}})^2] \leqslant 4(\hbar/E)^2.$$

By selecting $\hat{\boldsymbol{r}}^\intercal = (1,0,0)$ and $\hat{\boldsymbol{n}}^\intercal = (0,0,1)$ we can construct the scheme

$$\rho_0 = (\mathbb{I} + \sigma_x)/2, \; \mathcal{G} = (E/\hbar)\sigma_z$$

for which F_q is maximum. Hence,

$$\rho(t) = [\mathbb{I} + \cos(2Et/\hbar)\sigma_x + \sin(2Et/\hbar)\sigma_y]/2.$$

Asymptotically optimal POVM

From the analysis of $1/(\mu F_q)$ we can find

$$E(s_\pm) = (\mathbb{I} \pm \sigma_y)/2.$$

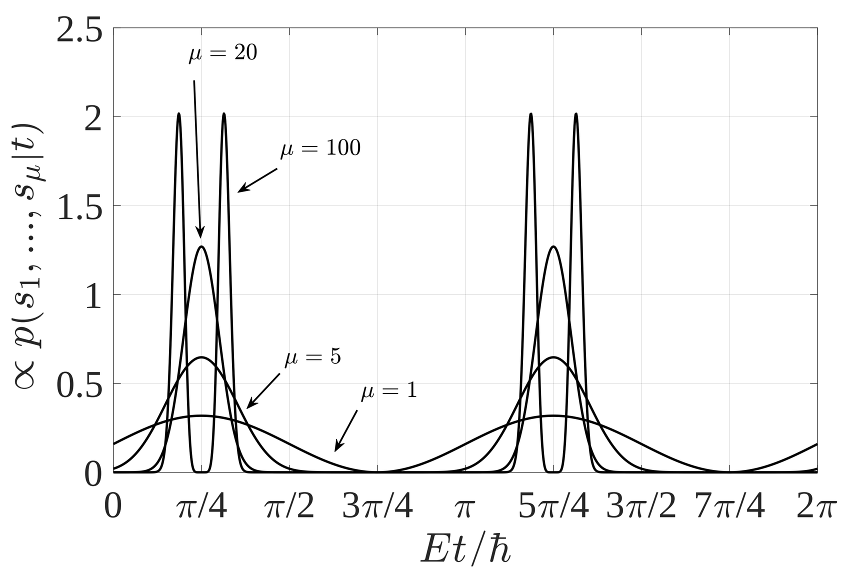
The qubit clock II: Bayesian results

Probabilities: the likelihood function

The Born rule applied to $E(s_\pm)$ and $\rho(t)$ gives us the probabilities

$$p(s_\pm|t) = \mathrm{Tr}[E(s_\pm)\rho(t)] = [1 \pm \sin(2Et/\hbar)]/2.$$

If we use these probabilities to simulate μ trials of our experiment, we find that



That is,

- $p(s_\pm|t) = p(s_\pm|t + P_k)$, with $P_k = k\pi\hbar/E$, and
- $p(s_\pm|t) = p(s_\pm|2A_k - t)$, with $A_k = \pi\hbar(2k+1)/(4E)$.

Hence, this scheme requires that the elapsed time t is known to lie within an interval of width $W_0 \leqslant \pi\hbar/(2E)$.

Available prior information

$$p(t) = 1/W_0, \; t \in [a_0, a_0 + W_0],$$

where $a_0 = \pi\hbar/(4E)$ and $W_0 = \pi\hbar/(2E)$.

Optimal single-shot quantum estimator and POVM

The optimal quantum estimator is

$$\tilde{T} = [\pi\hbar/(2E)](\mathbb{I} - 2\sigma_y/\pi^2),$$

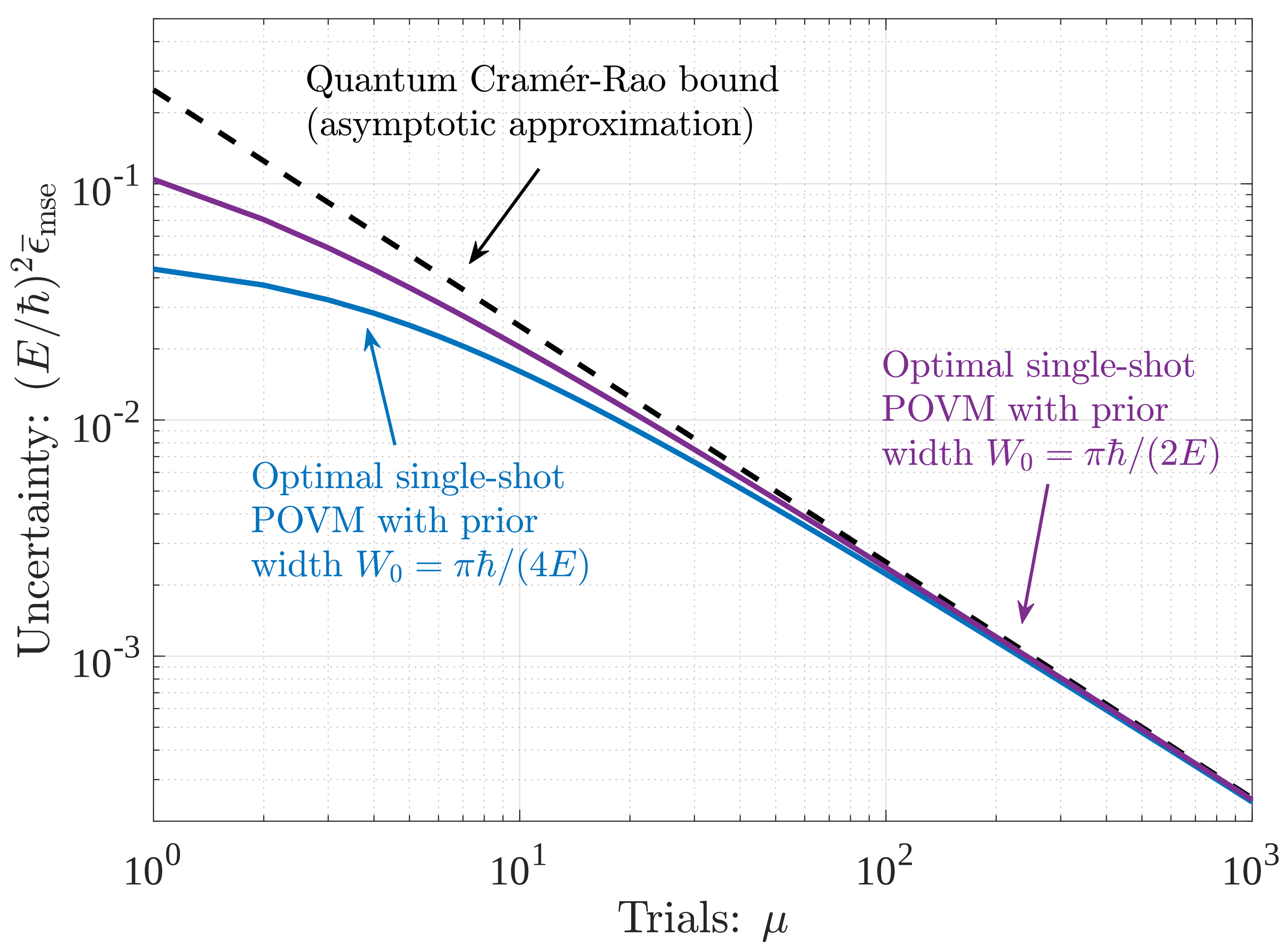
with estimates $s_\pm = [\pi\hbar/(2E)](1 \mp 2/\pi^2)$ and projectors $E(s_\pm) = (\mathbb{I} \pm \sigma_y)/2$. Thus the same POVM that is optimal for $\mu \gg 1$ is also optimal for $\mu = 1$.

Posterior probability

$$p(t|s_\pm) = [2E/(\pi\hbar)][1 \pm \sin(2Et/\hbar)].$$

This probability contains the information about the elapsed time t provided by an experiment with outcomes $\boldsymbol{s}$.

Time estimation with moderate prior information and limited data



Bayesian uncertainty for the estimation of the elapsed time t. We can observe how the low-μ performance of our scheme deviates significantly from the prediction of the asymptotic inequality. Furthermore, our method captures the crucial role that different states of prior information play in the regime of limited data.

Discussion and conclusions

- We have demonstrated how to apply the Bayesian methodology introduced in [1,2] for the estimation of elapsed time.
- We have studied how different amounts of prior information affect the estimation of the elapsed time in the regime of limited data.
- We have established the differences between $(\mu F_q)\bar{\epsilon}_{\mathrm{mse}} \gtrsim 1$ (see, e.g., [5]) and $\Delta\tilde{t}^2 + \bar{\epsilon}_{\mathrm{mse}} \geqslant \Delta t_p^2$ for the study of time in a quantum context. This suggests a potentially broader use of the latter inequality, which has the advantage of including prior information.
- Future work may explore possibilities such as the applicaton of this method to a realistic clock or the analysis of the differences and similarities between our work and other approaches [6,7].

References

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